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AN EXAMPLE FOR UNDERSTANDING NONLINEAR PREDICTION ALGORITHMS

by

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0. Summary

The purpose of this memorandum is to outline a relatively simple example which may serve as a basis for understanding advantages and disadvantages of prediction algorithms from conceptual, complexity, and accuracy viewpoints.

1. Introduction

The ability to predict the future response of a physical system is of extreme importance in both engineering and nonengineering problems. Quite often, our inability to predict accurately the future response of a system is due to:

(a) lack of precise equations of motion, and

(b) the application of unpredictable inputs to the system from the present time until the future time.

A class of problems in which prediction is of paramount importance is found in the defense area. Typically, in such problems one tracks a reentry vehicle via a radar. The radar observations are introduced into a tracking data processing algorithm (often called the extended Kalman filter) which has the ability to generate continuously in time, and on the basis of prior observations,

(a) estimates of the target position,

(b) estimates of the target velocity, and

(c) estimates of the target ballistic parameters.

In addition, to these estimates the tracking algorithm also generates continuously in time an error pseudo covariance matrix whose elements specify approximately the auto-and-cross-covariance functions of the estimation errors.

The prediction problem is fundamentally different. One is interested, on the basis of the measurements up to now, to predict the future trajectory of the target. In addition, one is often interested in the variations of the trajectory from the predicted one; such considerations can be used to define a "tube" about the predicted uncertainty which, in time, can be used to define a "footprint" about the predicted impact point.

It should be self-evident that accurate prediction of target motion is of extreme importance in the defense problem. First, accurate early impact prediction can be used to make a decision of the allocation of a battery of interceptors of intercept a given target. Second, accurate prediction of the target prior to impact is essential for successful interception. If at the predicted intercept time the target is far from its predicted location, it may be necessary for the interceptor to undergo violent acceleration maneuvers which may be impossible to carry out due to limitations on interceptor agility and maneuverability.

The current thinking on target trajectory prediction can be roughly described as follows: the instantaneous estimates of the target position, velocity, and ballistic parameters, as generated by the tracking algorithm, are used as initial conditions in the integration of the deterministic differential equations of motion that describe the target. The predicted target trajectory is then obtained by the integration (in general, numerical integration) of the target differential equations. This is the optimal thing to do if the target differential equations are linear.[1] However, there is no reason whatsoever to suspect that this is the best that one can do if the differential equations are significantly nonlinear. In point of fact, there is some suspicion that such a prediction algorithm may be sensitive to bias errors and there is some evidence that this is the case even in tracking problems with low observation data rates.[2,3]

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[1] See R.E. Kalman and R.S. Bucy "New Results in Linear Filtering and Prediction Theory", Trans. ASME, J. Basic Eng'g., Vol. 83, pp. 95-108, 1961.

[2] M. Athans, R.P. Wishner, and A. Bertolini, "Suboptimal State Estimation for Continuous-Time Nonlinear Systems from Discrete Noisy Measurements", IEEE Trans. on Autom. Control, Vol. AC-13, pp. 504-514, 1968.

[3] R.P. Wishner, J.A. Tabaczynski, and M. Athans, "A Comparison of Three Nonlinear Filters", Automatica, Vol. 5, pp. 487-496.

It is the opinion of the author that some fundamental understanding of the prediction problem for nonlinear systems is necessary. In the remainder of this memorandum, some remarks on the prediction problem will be made and some simulation studies will be presented.

2. Prediction for Linear Systems

In order to understand some of the techniques associated with prediction, let us consider the case of linear systems. Suppose that we have a linear, possibly time-varying, system whose state vector $\underline{x}(t)$ satisfies the differential equation

$$\dot{\underline{x}}(t) = \underline{A}(t)\underline{x}(t) + \underline{v}(t) \;\; ; \;\; \underline{x}(t_0) = \underline{x}_0 \tag{1}$$

We assume that the n x n matrix $\underline{A}(t)$ and the n-column vector $\underline{v}(t)$ are known and deterministic. Let $\underline{x}_0 = \underline{x}(t_0)$ be the true state of the system at time t_0. We suppose that $\underline{x}_0$ is a random vector which is Gaussian. Suppose that the mean and covariance matrix of $\underline{x}_0$ are known (this is enough to specify the Gaussian distribution) and that

$$E\left\{\underline{x}_0\right\} \triangleq \bar{\underline{x}}_0 \tag{2}$$

$$E\left\{(\underline{x}_0 \;\; \bar{\underline{x}}_0) \;\; (\underline{x}_0 - \bar{\underline{x}}_0)'\right\} \triangleq \underline{S}_0 \tag{3}$$

What we are interested in is to predict the value of $\underline{x}(t)$, for $t \geq t_0$, of the true state vector on the basis of the available probabilistic information. We shall let $\hat{\underline{x}}(t)$ denote the <u>predicted estimate</u> of $\underline{x}(t)$, $t > t_0$. We define the <u>prediction error</u> vector $\underline{e}(t)$ by

$$\underline{e}(t) \triangleq \underline{x}(t) - \hat{\underline{x}}(t) \tag{4}$$

Let us now restrict our attention to the <u>class of linear predictors</u>. In other words, we shall generate the predicted state $\hat{\underline{x}}(t)$ by solving the linear vector differential equation

$$\dot{\hat{\underline{x}}}(t) = \underline{F}(t)\hat{\underline{x}}(t) + \underline{w}(t) \;\; ; \;\; \hat{\underline{x}}(t_0) = \hat{\underline{x}}_0 \tag{5}$$

We remark that eq. (5) specifies the structure (linear) but not the "coefficients" of the prediction algorithm. To specify the prediction algorithm completely we must specify

$$\left.\begin{array}{l}\text{(a) the } n \times n \text{ matrix } \underline{F}(t) \\ \text{(b) the } n\text{-column vector } \underline{w}(t)\text{, and} \\ \text{(c) the initial estimate } \hat{\underline{x}}_0\end{array}\right\} \tag{6}$$

Thus, we have the freedom (even under the structural constraints) to choose the quantities in eq. (6) so that the predicted state $\hat{\underline{x}}(t)$ is in some sense "good".

One criterion of goodness is that of unbiasedness. We shall now show that if we impose the unbiasedness requirement, then the prediction algorithm (5) becomes completely specified in the sense that all the "free parameters" of eq. (6) are found in terms of known parameters.

The derivation proceeds as follows. We desire to have the error process (4) to have zero mean

$$E\left\{\underline{e}(t)\right\} = \underline{0} \text{ for all } t \geq t_0 \tag{7}$$

Hence,

$$\frac{d}{dt} E\left\{\underline{e}(t)\right\} = \underline{0} \text{ for all } t \geq t_0 \tag{8}$$

which under suitable assumptions implies

$$E\left\{\dot{\underline{e}}(t)\right\} = \underline{0} \text{ for all } t \geq t_0 \tag{9}$$

From eqs.(4),(1), and (5) we obtain

$$\dot{\underline{e}}(t) = \dot{\underline{x}}(t)-\dot{\hat{\underline{x}}}(t)$$

$$= \underline{A}(t)\underline{x}(t)+\underline{v}(t) - \underline{F}(t)\hat{\underline{x}}(t) - \underline{w}(t)$$

$$= \underline{A}(t)\underline{x}(t) + \underline{F}(t)\underline{x}(t) - \underline{F}(t)\underline{x}(t) - \underline{F}(t)\hat{\underline{x}}(t) + \underline{v}(t) - \underline{w}(t)$$

$$= \underline{F}(t)[\underline{x}(t) - \hat{\underline{x}}(t)] + [\underline{A}(t) - \underline{F}(t)]\underline{x}(t) + \underline{v}(t) - \underline{w}(t)$$

$$= \underline{F}(t)\underline{e}(t) + [\underline{A}(t) - \underline{F}(t)]\underline{x}(t) + \underline{v}(t) - \underline{w}(t) \tag{10}$$

By taking expectations of both sides of (10) we obtain

$$E\{\dot{\underline{e}}(t)\} = \underline{F}(t)E\{\underline{e}(t)\} + [\underline{A}(t)-\underline{F}(t)]E\{\underline{x}(t)\} + \underline{v}(t)-\underline{w}(t) \tag{11}$$

Since $E\{\underline{x}(t)\} \neq \underline{0}$, in general, the conditions (7) and (9) yield

$$\underline{F}(t) = \underline{A}(t) \tag{12}$$

$$\underline{v}(t) = \underline{w}(t) \tag{13}$$

Furthermore, the condition

$$E\{\underline{e}(t_0)\} = E\{\underline{x}(t_0)-\hat{\underline{x}}(t_0)\} = E\{\underline{x}_0\} -\hat{\underline{x}}_0 \tag{14}$$

yields

$$\hat{\underline{x}}_0 = \bar{\underline{x}}_0 \tag{15}$$

These equations imply that the predicted state $\hat{\underline{x}}(t)$ is generated as a solution of the equation

$$\dot{\hat{\underline{x}}}(t) = \underline{A}(t)\hat{\underline{x}}(t) + \underline{v}(t) \quad ; \quad \hat{\underline{x}}(t_0) = \bar{\underline{x}}_0 \tag{16}$$

This completely specifies the prediction algorithm.

The prediction error satisfies the differential equation

$$\dot{\underline{e}}(t) = \underline{A}(t)\underline{e}(t) \tag{17}$$

Let $\underline{S}(t)$ denote the prediction error covariance matrix; since $\underline{e}(t)$ has zero mean then

$$\underline{S}(t) = E\{\underline{e}(t)\underline{e}'(t)\} \tag{18}$$

To determine $\underline{S}(t)$ we differentiate eq. (20).

$$\dot{\underline{S}}(t) = \frac{d}{dt} E\{\underline{e}(t)\underline{e}'(t)\}$$

$$= E\{\dot{\underline{e}}(t)\underline{e}'(t)\} + E\{\underline{e}(t)\dot{\underline{e}}'(t)\}$$

$$= E\{\underline{A}(t)\underline{e}(t)\underline{e}'(t)\} + E\{\underline{e}(t)\underline{e}'(t)\underline{A}'(t)\}$$

$$= \underline{A}(t)E\{\underline{e}(t)\underline{e}'(t)\} + E\{\underline{e}(t)\underline{e}'(t)\}\underline{A}'(t)$$

$$= \underline{A}(t)\underline{S}(t) + \underline{S}(t)\underline{A}'(t) \tag{19}$$

But, in view of eqs. (15) and (3)

$$\underline{S}(t_0) = E\left\{\underline{e}(t_0)\underline{e}'(t_0)\right\} = E\left\{\left(\underline{x}_0-\hat{\underline{x}}_0\right)\left(\underline{x}_0-\hat{\underline{x}}_0\right)'\right\}$$

$$= E\left\{\left(\underline{x}_0-\bar{\underline{x}}_0 \quad \underline{x}_0-\bar{\underline{x}}_0\right)'\right\} = \underline{S}_0 \tag{20}$$

Hence, the prediction error covariance matrix is generated by the linear matrix differential equation

$$\dot{\underline{S}}(t) = \underline{A}(t)\underline{S}(t) + \underline{S}(t)\underline{A}'(t) \quad ; \quad \underline{S}(t_0) = \underline{S}_0 \tag{21}$$

Furthermore, in view of the linearity structure of the predictor, one can show that $\underline{e}(t)$ is Gaussian.

3. Prediction for Nonlinear Systems

The development of prediction algorithms for nonlinear systems is not as straightforward as that for linear systems. To illustrate the similarities and the differences involved we shall examine a specific, simple, first order nonlinear system.

Consider a system whose scalar variable $x(t)$ satisfies the nonlinear differential equation

$$\dot{x}(t) = a_2 x^2(t) + a_1 x(t) + a_0 + b(t) \tag{22}$$

where a_0, a_1, a_2 are known constants and $b(t)$ is a known time function

We assume that at $t = 0$, the initial state

$$x(0) \triangleq x_0 \tag{23}$$

is a Gaussian random variable with known mean $\bar{x}_0$, i.e.

$$E\, x_0 = \bar{x}_0 \tag{24}$$

and known variance Σ_0, i.e.

$$E(x_0 - \bar{x}_0)^2 = \Sigma_0 \tag{25}$$

Our objective is to generate a predicted estimate $\hat{x}(t)$ of the state $x(t)$ on the basis of the above information and in the absence of any additional measurements.

With no loss of generality we can fix the structure of the predictor to be

$$\dot{\hat{x}}(t) = a_2 \hat{x}^2(t) + a_1 \hat{x}(t) + a_0 + b(t) + \mu(t) \;;\; \hat{x}(0) = \hat{x}_0 \tag{26}$$

Since a_0, a_1, a_2 and $b(t)$ are assumed known, to completely specify the prediction algorithm (26) we must

$$\left.\begin{array}{l}\text{(a) specify } \mu(t) \text{ for } t \geq 0 \\ \text{(b) specify } \hat{x}_0\end{array}\right\} \tag{27}$$

We shall indicate below how the quantities in (27) can be found on the basis of demanding that the prediction error e(t)

$$e(t) \stackrel{\Delta}{=} x(t)-\hat{x}(t) \tag{28}$$

be of zero mean (conditioned on the knowledge of the statistics of x_0 only) for all $t \geq 0$.

Differentiating both sides of (28) we obtain

$$\dot{e}(t) = \dot{x}(t)-\dot{\hat{x}}(t) \tag{29}$$

Substituting (22) and (26) into (29) yields

$$\begin{aligned}\dot{e}(t) &= a_2x^2(t) + a_1x(t) + a_0 + b(t) \\ &\quad - a_2\hat{x}^2(t) - a_1\hat{x}(t) - a_0 - b(t) - \mu(t) \\ &= a_2x^2(t) - a_2\hat{x}^2(t)) + a_1e(t) - \mu(t) \\ &= a_2(e(t) + \hat{x}(t))^2 - a_2\hat{x}^2(t) + a_1e(t) - \mu(t) \\ &= a_2e^2(t) + 2a_2\hat{x}(t)e(t) + a_2\hat{x}^2(t) - a_2\hat{x}^2(t) + a_1e(t) - \mu(t)\end{aligned} \tag{30}$$

Hence, the prediction error satisfies the differential equation

$$\dot{e}(t) = a_2e^2(t) + \left[2a_2\hat{x}(t) + a_1\right] e(t)-\mu(t) \tag{31}$$

subject to the initial condition

$$e_0 \triangleq e(0) = x_0 - \hat{x}_0 \tag{32}$$

In order to have

$$E\left\{e_0\right\} = 0 \tag{33}$$

it follows that the initial condition of the predictor (26) is

$$\hat{x}_0 = E\left\{x_0\right\} = \bar{x}_0 \tag{34}$$

Hence, under our assumption, the initial error e_0 is a Gaussian random variable with zero mean and variance

$$E\left\{{e_0}^2\right\} = E\left\{\left(x_0 - \bar{x}_0\right)^2\right\} = \Sigma_0 \text{ (known)} \tag{35}$$

Next, we demand that

$$E\left\{e(t)\right\} = 0 \quad \text{for all } t \geq 0 \tag{36}$$

This implies that

$$\frac{d}{dt} E\{e(t)\} = E\{\dot{e}(t)\} = 0 \quad \text{for all } t \geq 0 \tag{37}$$

Taking expectation of both sides of (31) yields

$$E\left\{\dot{e}(t)\right\} = a_2 E\left\{e^2(t)\right\} + \left[2a_2\hat{x}(t) + a_1\right] E\left\{e(t)\right\} - \mu(t) \tag{38}$$

Hence, from (36), (37), (38) we deduce that

$$\mu(t) = a_2 E\{e^2(t)\} \tag{39}$$

Let

$$S_2(t) \triangleq E\{e^2(t)\} \tag{40}$$

We note that $S_2(t)$ is the second moment of $e(t)$; it is indeed the variance of $e(t)$ if $E\left\{e(t)\right\} = 0$. Hence, the prediction algorithm (26) is

$$\dot{\hat{x}}(t) = a_2\hat{x}^2(t) + a_1\hat{x}(t) + a_0 + b(t) + a_2S_2(t) \; ; \hat{x}(0) = \bar{x}_0 \tag{41}$$

The error differential equation is

$$\dot{e}(t) = a_2e^2(t) + \left[2a_2\hat{x}(t) + a_1\right] e(t) - a_2S_2(t) \tag{42}$$

Hence to generate $x(t)$ we must know $S_2(t)$. This motivates us to obtain the differential equation for $S_2(t)$.

For reasons that will soon become apparent we define all the error moments by

$$\left.\begin{aligned} S_2(t) &= E\left\{e^2(t)\right\} \\ S_3(t) &= E\left\{e^3(t)\right\} \\ &\cdots\cdots\cdots\cdots \\ S_k(t) &= E\left\{e^k(t)\right\} \end{aligned}\right\} \tag{43}$$

To determine the algorithm that generates $S_2(t)$ we differentiate both sides of (40).

$$\dot{S}_2(t) = \frac{d}{dt} E\left\{e^2(t)\right\} = E\left\{\frac{d}{dt} e^2(t)\right\} = E\left\{2e(t)\dot{e}(t)\right\} \tag{44}$$

Substituting the error differential equation (42) into (44) we deduce

$$\begin{aligned} \dot{S}_2(t) &= E\left\{2e(t)\left[a_2e^2(t) + \left[2a_2\hat{x}(t) + a_1\right] e(t) - a_2S_2(t)\right]\right\} \\ &= 2a_2E\left\{e^3(t)\right\} + 2\left[2a_2\hat{x}(t) + a_1\right] E\left\{e^2(t)\right\} - 2a_2S_2(t)E\{e(t)\} \end{aligned} \tag{45}$$

From (36) and (43) we conclude that

$$\dot{S}_2(t) = 2\left[2a_2\hat{x}(t) + a_1\right] S_2(t) + 2a_2S_3(t) \tag{46}$$

with initial condition, in view of (36) and (40)

$$S_2(0) = \Sigma_0 \tag{47}$$

So to specify $S_2(t)$ we need $S_3(t)$. Since

$$S_3(t) \triangleq E\left\{e^3(t)\right\} \tag{48}$$

it follows that

$$\begin{aligned}\dot{S}_3(t) &= \frac{d}{dt} E\left\{e^3(t)\right\} = E\left\{3e^2(t)\dot{e}(t)\right\} \\ &= E\left\{3e^2(t)\left[a_2e^2(t) + \left[2a_2\hat{x}(t) + a_1\right] e(t)-a_2\ S_2(t)\right]\right\} \\ &= 3a_2E\left\{e^4(t)\right\} + 3\left[2a_2\hat{x}(t) + a_1\right] E\ e^3(t) - 3a_2S_2(t)E\left\{e^2(t)\right\}\end{aligned} \tag{49}$$

or

$$\dot{S}_3(t) = 3\left[2a_2\hat{x}(t) + a_1\right] S_3(t) - 3a_2S_2{}^2(t) + 3a_2S_4(t) \tag{50}$$

Since e_0 is a Gaussian random variable with zero mean and variance $\sum_0$ its moments are defined by

$$\left.\begin{aligned} S_2(0) &= \Sigma_0 \\ S_3(0) &= 0 \\ S_4(0) &= 3\Sigma_0{}^2 \\ S_5(0) &= 0 \\ S_6(0) &= 15\ \Sigma_0{}^3 \end{aligned}\right\}$$

(Equation 51 continued on next page)

and, in general,

$$S_k(0) = \begin{cases} 0 & \text{for } k \text{ odd} \\ (k-1)\ (k-3)\ (k-5)\ldots(1)\,\Sigma^{\frac{k}{2}} & \text{for } k \text{ even} \end{cases} \tag{51}$$

Hence, the initial condition to (50) is

$$S_3(0) = 0 \tag{52}$$

The pattern now becomes apparent; to compute $S_2(t)$ we need $S_3(t)$; to compute $S_3(t)$ we need $S_4(t)$ and so on. In general,

$$\frac{d}{dt} S_k(t) = \frac{d}{dt} E\left\{e^k(t)\right\} = E\left\{k\, e^{k-1}(t)\dot{e}(t)\right\}$$

$$= E\left\{k\, e^{k-1}(t)\left[a_2 e^2(t) + \left[2a_2\hat{x}(t) + a_1\right] e(t) - a_2 S_2(t)\right]\right\}$$

$$= k\left[2a_2\hat{x}(t) + a_1\right] E\left\{e^k(t)\right\} - a_2 k\, S_2(t)\, E\left\{e^{k-1}(t)\right\}$$

$$+\ a_2 k\, E\left\{e^{k+1}(t)\right\} \tag{53}$$

Hence

$$\dot{S}_k(t) = k\left[2a_2\hat{x}(t) + a_1\right] S_k(t) - a_2 k S_2(t) S_{k-1}(t) + a_2 k S_{k+1}(t) \tag{54}$$

with initial condition given by (51).

An examination of the above equations leads to the following conclusion: in order to generate a predictor $\hat{x}(t)$ such that the prediction error $e(t) = x(t) - \hat{x}(t)$ has zero mean one must solve an "infinite set" of ordinary, nonlinear differential equations as defined by eqs. (42), (47), (51), and (55).

The structure of the equations, however, suggested suboptimal prediction algorithms by simply ignoring the effect of the next highest moment $S_{k+1}(t)$ upon the differential equation for $S_k(t)$. This leads to the following class of prediction algorithms in order of increasing complexity.

<u>1st Order Prediction Algorithm</u>

$\hat{x}(t)$ is generated by

$$\dot{\hat{x}}(t) = a_2\hat{x}^2(t) + a_1\hat{x}(t) + a_0 + b(t) \;\; ; \;\; \hat{x}(0) = \bar{x}_0$$

<u>2nd Order Prediction Algorithm</u>

$\hat{x}(t)$ is generated by

$$\dot{\hat{x}}(t) = a_2\hat{x}^2(t) + a_1\hat{x}(t) + a_0 + b(t) + a_2S_2(t) \;\; ; \;\; \hat{x}(0) = \bar{x}_0$$

$$\dot{S}_2(t) = 2\left[2a_2x(t) + a_1\right]S_2(t) \;\; ; \;\; S_2(0) = \Sigma_0$$

<u>3rd order Prediction Algorithm</u>

$$\dot{\hat{x}}(t) = a_2\hat{x}^2(t) + a_1\hat{x}(t) + a_0 + b(t) + a_2S_2(t) \;\; ; \;\; \hat{x}(0) = \bar{x}_0$$

$$\dot{S}_2(t) = 2\left[2a_2\hat{x}(t) + a_1\right]S_2(t) + 2a_2S_3(t) \;\; ; \;\; S_2(0) = \Sigma_0$$

$$\dot{S}_3(t) = 3\left[2a_2\hat{x}(t) + a_1\right]S_3(t) - 3a_2S_2(t)S_2(t) \;\; ; \;\; S_3(0) = 0$$

<u>4th Order Prediction Algorithm</u>

$$\dot{\hat{x}}(t) = a_2\hat{x}^2(t) + a_1\hat{x}(t) + a_0 + b(t) + a_2S_2(t) \;\; ; \;\; \hat{x}(0) = \bar{x}_0$$

$$\dot{S}_2(t) = 2\left[2a_2\hat{x}(t) + a_1\right]S_2(t) + 2a_2S_3(t) \;\; ; \;\; S_2(0) = \Sigma_0$$

$$\dot{S}_3(t) = 3\left[2a_2\hat{x}(t) + a_1\right]S_3(t) - 3a_2S_2(t)S_2(t) + 3a_2S_4(t) \;\; ; \;\; S_3(0) = 0$$

$$\dot{S}_4(t) = 4\left[2a_2\hat{x}(t) + a_1\right] S_4(t) - 4a_2 S_2(t)\, S_3(t) \quad ; \quad S_4(0) = 3\Sigma_0^{\;2}$$

<u>5th Order Prediction Algorithm</u>

$$\dot{\hat{x}}(t) = a_2\hat{x}^2(t) + a_1\hat{x}(t) + a_0 + b(t) + a_2 S_2(t) \quad ; \quad \hat{x}(0) = \overline{x}_0$$

$$\dot{S}_2(t) = 2\left[2a_2\hat{x}(t) + a_1\right] S_2(t) + 2a_2 S_3(t) \quad ; \quad S_2(0) = \Sigma_0$$

$$\dot{S}_3(t) = 3\left[2a_2\hat{x}(t) + a_1\right] S_3(t) - 3a_2 S_2(t)\, S_2(t) + 3a_2 S_4(t) \quad ; \quad S_3(0) = 0$$

$$\dot{S}_4(t) = 4\left[2a_2\hat{x}(t) + a_1\right] S_4(t) - 4a_2 S_2(t) S_3(t) + 4a_2 S_5(t) \quad ; \quad S_4(0) = 3\Sigma_0^{\;2}$$

$$\dot{S}_5(t) = 5\left[2a_2\hat{x}(t) + a_1\right] S_5(t) - 5a_2 S_2(t) S_4(t) \quad ; \quad S_5(0) = 0$$

and so on

4. Numerical Results

In this section we present numerical results comparing six predictors. The numerical results are contained in Tables 1 through 6.

The following values were used,

$a_0 = 8$

$a_1 = -2$

$a_2 = -1$

$b(t) = 0$

$\overline{x}_0 = 10,\ \Sigma_0 = 2$

From the numerical results presented (and others not shown here) we conclude that the predicted state $\hat{x}(t)$ changes only in the third significant figure between the 1st and 2nd order predictions. The second moment, however, is somewhat more sensitive.

TABLE 1

1st Order Predictor

TIME	$\hat{x}(t)$
0.0	1.00E 01
0.05	6.40E 00
0.10	4.74E 00
0.15	3.82E 00
0.20	3.25E 00
0.25	2.88E 00
0.30	2.63E 00
0.35	2.45E 00
0.40	2.33E 00
0.45	2.24E 00
0.50	2.18E 00
0.55	2.13E 00
0.60	2.10E 00
0.65	2.07E 00
0.70	2.05E 00
0.75	2.04E 00
0.80	2.03E 00
0.85	2.02E 00
0.90	2.02E 00
0.95	2.01E 00
1.00	2.01E 00
1.05	2.01E 00
1.10	2.00E 00
1.15	2.00E 00

TABLE 2

2nd Order Predictor

TIME	$\hat{x}(t)$	$S_2(t)$
0.0	1.00E 01	2.00E 00
0.05	6.38E 00	3.36E-01
0.10	4.72E 00	9.24E-02
0.15	3.80E 00	3.25E-02
0.20	3.24E 00	1.32E-02
0.25	2.87E 00	5.89E-03
0.30	2.62E 00	2.79E-03
0.35	2.45E 00	1.38E-03
0.40	2.33E 00	7.00E-04
0.45	2.24E 00	3.63E-04
0.50	2.17E 00	1.91E-04
0.55	2.13E 00	1.02E-04
0.60	2.09E 00	5.47E-05
0.65	2.07E 00	2.96E-05
0.70	2.05E 00	1.60E-05
0.75	2.04E 00	8.72E-06
0.80	2.03E 00	4.75E-06
0.85	2.02E 00	2.60E-06
0.90	2.02E 00	1.42E-06
0.95	2.01E 00	7.77E-07
1.00	2.01E 00	4.26E-07
1.05	2.01E 00	2.33E-07
1.10	2.00E 00	1.28E-07
1.15	2.00E 00	7.01E-08
1.20	2.00E 00	3.84E-08
1.25	2.00E 00	2.11E-08
1.30	2.00E 00	1.16E-08

TABLE 3

3rd Order Predictor

TIME	$\hat{x}(t)$	$S_2(t)$	$S_3(t)$
0.0	1.00E 01	2.00E 00	0.0
0.05	6.38E 00	3.34E-01	2.65E-02
0.10	4.72E 00	9.11E-02	5.55E-03
0.15	3.80E 00	3.19E-02	1.36E-03
0.20	3.24E 00	1.29E-02	3.84E-04
0.25	2.87E 00	5.75E-03	1.20E-04
0.30	2.62E 00	2.72E-03	4.06E-05
0.35	2.45E 00	1.34E-03	1.44E-05
0.40	2.33E 00	6.80E-04	5.30E-06
0.45	2.24E 00	3.53E-04	2.00E-06
0.50	2.17E 00	1.86E-04	7.72E-07
0.55	2.13E 00	9.89E-05	3.02E-07
0.60	2.09E 00	5.31E-05	1.19E-07
0.65	2.07E 00	2.87E-05	4.75E-08
0.70	2.05E 00	1.55E-05	1.90E-08
0.75	2.04E 00	8.45E-06	7.63E-09
0.80	2.03E 00	4.61E-06	3.08E-09
0.85	2.02E 00	2.52E-06	1.24E-09
0.90	2.02E 00	1.38E-06	5.03E-10
0.95	2.01E 00	7.53E-07	2.04E-10
1.00	2.01E 00	4.13E-07	8.26E-11
1.05	2.01E 00	2.26E-07	3.35E-11
1.10	2.00E 00	1.24E-07	1.36E-11
1.15	2.00E 00	6.79E-08	5.53E-12
1.20	2.00E 00	3.73E-08	2.25E-12
1.25	2.00E 00	2.04E-08	9.12E-13
1.30	2.00E 00	1.12E-08	3.71E-13

TABLE 4

4th Order Predictor

TIME	$\hat{x}(t)$	$S_2(t)$	$S_3(t)$	$S_4(t)$
0.0	1.00E 01	2.00E 00	0.0	1.20E 01
0.05	6.38E 00	3.40E-01	-5.27E-02	3.36E-01
0.10	4.72E 00	9.48E-02	-1.10E-02	2.51E-02
0.15	3.80E 00	3.37E-02	-2.68E-03	3.09E-03
0.20	3.24E 00	1.38E-02	-7.57E-04	5.09E-04
0.25	2.87E 00	6.17E-03	-2.37E-04	1.01E-04
0.30	2.62E 00	2.93E-03	-7.97E-05	2.25E-05
0.35	2.45E 00	1.45E-03	-2.83E-05	5.48E-06
0.40	2.33E 00	7.39E-04	-1.04E-05	1.42E-06
0.45	2.24E 00	3.84E-04	-3.93E-06	3.81E-07
0.50	2.17E 00	2.03E-04	-1.51E-06	1.06E-07
0.55	2.13E 00	1.08E-04	-5.91E-07	2.99E-08
0.60	2.09E 00	5.80E-05	-2.34E-07	8.63E-09
0.65	2.07E 00	3.13E-05	-9.29E-08	2.51E-09
0.70	2.05E 00	1.70E-05	-3.72E-08	7.30E-10
0.75	2.04E 00	9.24E-06	-1.49E-08	2.19E-10
0.80	2.03E 00	5.04E-06	-6.02E-09	6.50E-11
0.85	2.02E 00	2.75E-06	-2.43E-09	1.94E-11
0.90	2.02E 00	1.50E-06	-9.84E-10	5.80E-12
0.95	2.01E 00	3.24E-07	-3.99E-10	1.74E-12
1.00	2.01E 00	4.51E-07	-1.62E-10	5.21E-13
1.05	2.01E 00	2.47E-07	-6.56E-11	1.56E-13
1.10	2.00E 00	1.35E-07	-2.66E-11	4.70E-14
1.15	2.00E 00	7.43E-08	-1.08E-11	1.41E-14
1.20	2.00E 00	4.07E-08	-4.39E-12	4.24E-15
1.25	2.00E 00	2.23E-08	-1.78E-12	1.28E-15
1.30	2.00E 00	1.23E-08	-7.24E-13	3.84E-16

TABLE 5

5th Order Predictor

TIME	$\hat{x}(t)$	$S_2(t)$	$S_3(t)$	$S_4(t)$	$S_5(t)$
0.0	1.00E 01	2.00E 00	0.0	1.29E 01	0.0
0.05	6.38E 00	3.40E-01	-5.22E-02	3.29E-01	4.44E-02
0.10	4.72E 00	9.48E-02	-1.08E-02	2.40E-02	2.50E-03
0.15	3.80E 00	3.37E-02	-2.60E-03	2.90E-03	2.20E-04
0.20	3.24E 00	1.38E-02	-7.29E-04	4.71E-04	2.52E-05
0.25	2.87E 00	6.17E-03	-2.27E-04	9.24E-05	3.52E-06
0.30	2.62E 00	2.93E-03	-7.61E-05	2.05E-05	5.62E-07
0.35	2.45E 00	1.45E-03	-2.69E-05	4.07E-06	9.84E-08
0.40	2.33E 00	7.38E-04	-9.89E-06	1.28E-06	1.84E-08
0.45	2.24E 00	3.84E-04	-3.73E-06	3.43E-07	3.61E-09
0.50	2.17E 00	2.02E-04	-1.44E-06	9.51E-08	7.33E-10
0.55	2.13E 00	1.08E-04	-5.61E-07	2.69E-08	1.53E-10
0.60	2.09E 00	5.79E-05	-2.22E-07	7.75E-09	3.24E-11
0.65	2.07E 00	3.13E-05	-8.81E-08	2.26E-09	6.95E-12
0.70	2.05E 00	1.70E-05	-3.53E-08	6.63E-10	1.51E-12
0.75	2.04E 00	9.22E-06	-1.42E-08	1.06E-10	3.30E-13
0.80	2.03E 00	5.03E-06	-5.71E-09	5.82E-11	7.25E-14
0.85	2.02E 00	2.75E-06	-2.31E-09	1.74E-11	1.60E-14
0.90	2.02E 00	1.50E-06	-9.33E-10	5.19E-12	3.54E-15
0.95	2.01E 00	8.22E-07	-3.78E-10	1.56E-12	7.85E-16
1.00	2.01E 00	4.50E-07	-1.53E-10	4.67E-13	1.74E-16
1.05	2.01E 00	2.47E-07	-6.22E-11	1.40E-13	3.87E-17
1.10	2.00E 00	1.35E-07	-2.52E-11	4.21E-14	8.61E-18
1.15	2.00E 00	7.42E-08	-1.02E-11	1.26E-14	1.92E-18
1.20	2.00E 00	4.07E-08	-4.16E-12	3.80E-15	4.27E-19
1.25	2.00E 00	2.23E-08	-1.69E-12	1.14E-15	9.51E-20
1.30	2.00E 00	1.22E-08	-6.86E-13	3.44E-16	2.12E-20

TABLE 6

6th Order Predictor

TIME	$\hat{x}(t)$	$S_2(t)$	$S_3(t)$	$S_4(t)$	$S_5(t)$	$S_6(t)$
0.0	1.00E 01	2.00E 00	0.0	1.20E 01	0.0	1.20E 02
0.05	6.38E 00	3.40E-01	-5.49E-02	3.64E-01	-1.76E-01	5.56E-01
0.10	4.72E 00	9.49E-02	-1.20E-02	2.96E-02	-9.96E-03	1.12E-02
0.15	3.80E 00	3.38E-02	-3.01E-03	3.84E-03	-8.48E-04	4.78E-04
0.20	3.24E 00	1.38E-02	-8.68E-04	6.56E-04	-9.67E-05	3.17E-05
0.25	2.87E 00	6.20E-03	-2.75E-04	1.33E-04	-1.34E-05	2.77E-06
0.30	2.62E 00	2.95E-03	-9.36E-05	3.03E-05	-2.13E-06	2.92E-07
0.35	2.45E 00	1.46E-03	-3.34E-05	7.46E-06	-3.73E-07	3.50E-08
0.40	2.33E 00	7.43E-04	-1.24E-05	1.94E-06	-6.96E-08	4.58E-09
0.45	2.24E 00	3.86E-04	-4.69E-06	5.26E-07	-1.36E-08	6.38E-10
0.50	2.17E 00	2.04E-04	-1.81E-06	1.47E-07	-2.77E-09	9.31E-11
0.55	2.13E 00	1.09E-04	-7.09E-07	4.17E-08	-5.75E-10	1.40E-11
0.60	2.09E 00	5.83E-05	-2.80E-07	1.20E-08	-1.22E-10	2.17E-12
0.65	2.07E 00	3.15E-05	-1.12E-07	3.52E-09	-2.62E-11	3.41E-13
0.70	2.05E 00	1.71E-05	-4.47E-08	1.03E-09	-5.69E-12	5.44E-14
0.75	2.04E 00	9.29E-06	-1.80E-08	3.06E-10	-1.24E-12	8.75E-15
0.80	2.03E 00	5.07E-06	-7.25E-09	9.12E-11	-2.73E-13	1.42E-15
0.85	2.02E 00	2.77E-06	-2.93E-09	2.72E-11	-6.02E-14	2.31E-16
0.90	2.02E 00	1.51E-06	-1.19E-09	8.14E-12	-1.33E-14	3.77E-17
0.95	2.01E 00	8.28E-07	-4.81E-10	2.44E-12	-2.95E-15	6.19E-18
1.00	2.01E 00	4.54E-07	-1.95E-10	7.32E-13	-6.56E-16	1.02E-18
1.05	2.01E 00	2.49E-07	-7.91E-11	2.20E-13	-1.46E-16	1.67E-19
1.10	2.00E 00	1.36E-07	-3.21E-11	6.60E-14	-3.24E-17	2.75E-20
1.15	2.00E 00	7.47E-08	-1.30E-11	1.98E-14	-7.21E-18	4.54E-21
1.20	2.00E 00	4.10E-08	-5.29E-12	5.96E-15	-1.61E-18	7.48E-22
1.25	2.00E 00	2.25E-08	-2.15E-12	1.79E-15	-3.58E-19	1.23E-22
1.30	2.00E 00	1.23E-08	-8.73E-13	5.40E-16	-7.97E-20	2.04E-23